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On Some Non Newtonian Fluids

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Abstract: We consider the class of Rivlin Ericksen fluids of complexity 2 satisfying the Clausius–Duhem inequality. For strictly isolated fluids, we show that, if the Helmholtz free energy is not minimum in equilibrium, then the initial perturbation is (under some conditions) not only amplified, but also that it increases indefinitely. These results extend the analysis given by Fosdick Rajagopal in 1982 for the second grade fluids, which are a special case of Rivlin Ericksen fluids of complexity 2.

Keywords: non Newtonian fluids; differential-type fluids; fluids of grade 2; Clausius-Duhem inequality; Helmholtz free energy

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1 Introduction

It is well-known that even if the Navier-Stokes equations are considered to be a good model for many fluids in the nature, there are many cases for which these equations can not apply. More and more fluids, existing in nature or created by industries (for example, some cristal liquids, bitums or oils), exhibit what called "non Newtonian" properties as the Weisenberg effect (climbing on a rod), the die-swell, a.s.o., properties observed and measured in practice. Fluids governed by the Navier-Stokes equations can not have such behaviors. Indeed, these properties are described by the viscometric functions T_{12}, $T_{11}-T_{22}$ et $T_{22}-T_{33}$, where $T=(T_{i,j})_{1\leq i,j\leq 3}$ denotes the stress tensor. Consider a simple shear flow, i.e., the motion of a fluid between two walls, one fixed, the other one moving with the speed kx, (with k being the shear stress). In the Newtonian case, it is easily seen that T_{12} is linear in k, and moreover, $T_{11}-T_{22}=T_{22}-T_{33}=0$.

In the non Newtonian case, it is observed that T_{12} is generally not linear in k, $T_{11}-T_{22}\neq 0$ and $T_{22}-T_{33}\neq 0$. So, for non Newtonian fluids, one has to introduce other constitutive laws than the law leading to the Navier-Stokes equations (modeling the Newtonian fluids)

$$T=-pI+2\mu D,$$

where u is the velocity field, p is the pressure, $\mu\geq 0$ denotes the viscosity of the fluid, I is the unit tensor and D is the symmetrized rate of deformations tensor

$$D=2(L+L^t)=\frac{\nabla u+(\nabla ut)^t}{2},\quad \text{with}\quad L=\nabla u.$$

The non Newtonian theory is developed in several different directions. Let us mention some examples of non Newtonian fluids for which mathematical studies have been done: visco-elastic

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fluids (cf. for instance, Guillopé and Saut[1] and the references herein), Reiner-Rivlin fluids (cf. for instance, Duvaut and Lions[2], Ladyzhenkaya[1] Cioranescu[3]) and the differential-type fluids (cf. for instance, Bernard[4], Cioranescu and Ouazar[5], Cioranescu and Girault[6]).

The constitutive law for differential-type fluids (also called Rivlin-Ericksen fluids) was introduced by Rivlin[7] (see also Truesdell[8]). In this theory it is supposed that the material is homogeneous and isotropic, and that the stress tensor depends only on the gradient of the velocity via the Rivlin-Ericksen tensors, defined reccurently as follows

$$A_1 = L + L^t = 2D, \quad \cdots, \quad A_n = \frac{\mathrm{d}}{\mathrm{d}t}A_{n-1} + L^T A_{n-1} + A_{n-1}L.$$

Then, the constitutive law for a differential fluid of complexity n is given (see Noll and Truesdell[9]) by

$$T = -pI + F(A_1, A_2, \cdots, A_n),$$

where F is an isotropic function, i.e., satisfying for any orthogonal matrix Q,

$$QF(A_1, A_2, \cdots, A_n)Q^T = F(QA_1Q^T, QA_2Q^T, \cdots, QA_nQ^T).$$

The general form of an isotropic function F is given in Wang[10]. For example, for a fluid of complexity two with density $\rho > 0$,

$$\begin{aligned} F \ = \ & \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 + \alpha_3 A_2^2 + \alpha_4(A_1A_2 + A_2A_1) + \alpha_5(A_1^2A_2 + A_2A_1^2) \\ & +\alpha_6(A_2^2A_1 + A_1A_2^2) + \alpha_7(A_1^2A_2^2 + A_2^2A_1^2), \end{aligned}$$

where $\mu = \mu(\rho, A_1, A_2)$ and $\alpha_i = \alpha_i(\rho, A_1, A_2)$ with $1 \le i \le 7$, depend on the following functions, all isotropic: ρ, $\mathrm{tr}A_1$, $\mathrm{tr}A_1^2$, $\mathrm{tr}A_1^3$, $\mathrm{tr}A_2$, $\mathrm{tr}A_2^2$, $\mathrm{tr}A_2^3$, $\mathrm{tr}A_1A_2$, $\mathrm{tr}A_1^2A_2$, $\mathrm{tr}A_2^2A_1$, $\mathrm{tr}A_1^2A_2^2$. The function F can also depend on the temperature θ. When F is a polynomial of degree n, the fluid is called of grade n. In this framework, the constitutive law of a fluid of grade 1 is

$$T = -pI + \mu A_1$$

which in fact, is the definition of a Newtonian fluid. For a fluid of grade 2, one has

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2. \tag{1}$$

Taking into consideration the dissipation Clausius-Duhem inequality, a thermodynamical study due to Dunn et Fosdick[11], proved that the coefficients in (1) have to satisfy the following conditions

$$\mu \ge 0, \tag{2}$$

$$\alpha_1 + \alpha_2 = 0. \tag{3}$$

Suppose moreover, that the Helmholtz free energy is minimum at the equilibrium, one also has

$$\alpha_1 \ge 0. \tag{4}$$

Under these conditions, the fluid is stable. Dunn et Fosdick[11] and Fosdick et Rajagopal[12] have also proved that if $\alpha_1 < 0$ and $\alpha_1 + \alpha_2 = 0$, then the motion of the fluid is asymptotically unbounded. Such a behavior has never been observed experimentally, it is why one may consider that the only admissible fluids of grade 2 could be those where (2)-(4) are satisfied.

The positive sign of α_1 gives rise to many controversial discussions in the litterature (see for instance, Truesdell[8], Dunn and Rajagopal[13]). It is why, weaker restrictions than $\alpha_1 \geq 0$ have been introduced later on. In particular, Dunn[14] studied the case where the coefficients in (1), depend on the temperature

$$T = -pI + \mu(\theta, A_1)A_1 + \alpha_1(\theta, A_1)A_2 + \alpha_2(\theta, A_1)A_1^2. \tag{5}$$

Under the restrictive hypothesis that the free energy is minimum at the equilibrium, he showed that

$$\alpha_1(\theta, A_1) = \alpha_1(\theta, \mathrm{tr}A_1^2), \tag{6}$$

and

$$\int_0^z \alpha_1(\theta, \xi)d\xi \geq 0, \quad \forall\, z \in (0, \varepsilon), \quad \varepsilon > 0. \tag{7}$$

This class of fluids, under appropriate hypotheses, has reasonable mechanical and dynamical stability properties. Conditions (6) et (7) are less restrictive than (4) in the constant case.

Our aim here is to study the instability and the asymptotic behavior of the fluids defined by the law (5) and to weaken the restrictions on the coefficients α_1 et α_2 mentioned above. Following the ideas of Dunn and Fosdick[11] and of Fosdick and Rajagopal[12], we do not require that the free energy of fluids defined by (5), be minimum at equilibrium (as is the case in Dunn[14]).

Theorem 2.3 below shows (under some additional hypotheses), that if μ is sufficiently large and if $\alpha_2 \in L^{3/2}(\Omega)$, then any initial perturbation is not only amplified, but it also increases indefinitely. This result generalizes Theorem 1 of Fosdick et Rajagopal[12], where α_1 and α_2 were constants.

Proposition 2.5 describes the asymptotic behavior of the solution as $t \to +\infty$. We show that if μ is sufficiently large, then

$$\liminf_{t\to+\infty} \left(\sup |A_1|(t)\right) \neq 0,$$

which is contrary to the usual Newtonian behavior. This result generalizes Theorem 2 from Fosdick et Rajagopal[12].

Let now consider the cas where $\alpha_1 + \alpha_2 \equiv 0$ on $\Omega \times [0, +\infty)$. Theorems 2.6 and 2.8 below extend to the non constant case of α_1 and α_2, the corresponding results from [11] and Theorem 17 of [12]. They exhibit an anomaly of the type $\|u(t)\|^2 \geq c_1 N(0) \exp \lambda t$, for $\lambda \in [0, 2\mu/c_1]$, where N is defined by (11) and c_1 is given by (9) below.

2 Setting of the problem and main results

Let $\Omega \subset \mathbf{R}^3$ be a bounded rigid connected domain with a boundary sufficiently smmoth. It is filled with a homogeneous incompressible fluid obeying to (5). We suppose that we are given

an initial data at time $t = 0$, that on the boundary $\partial\Omega$ one has an adherence condition and that the exterior field of forces derives from a potential.

The flow of this fluid is described by the system

$$\begin{cases} \operatorname{div} T + \rho f = \rho \dfrac{\partial u}{\partial t}, & \text{in } \ \Omega \times [0, +\infty), \\ \operatorname{div} u = 0, & \text{in } \ \Omega \times [0, +\infty), \\ u(x,t) = 0, & \text{for } \ x \in \partial\Omega, \quad \forall\, t \geq 0, \\ u(x,0) = u_0(x), & \text{for } \ x \in \Omega, \end{cases} \tag{8}$$

where $\rho > 0$ is the density of the fluid.

In the sequel we use the following notation:

$|A| = \sqrt{\mathrm{tr} AA^t}$ for any tensor A, tr stands for the trace of the tensor;

C_Ω is the Poincaré constant associated to Ω;

$\mathcal{T}_s$ is the set of symmetric tensors while $\mathcal{T}_s^0$ the set of symmetric tensors with zero trace.

To study the field of velocities u defined in problem (8), we will be only interested in regular motions, the temperature will not play any role since the field of temperatures is considered uniform in the domain Ω and constant with respect to any $t \geq 0$.

Throughout the paper we make the following assumptions:

1) The density ρ is a positive constante and to simplify, we set $\rho = 1$;
2) The viscosity μ is a strictly positive constant;
3) The response functions α_1 and α_2 are sufficiently smooth on $\Omega \times [0, +\infty)$ and such that

$$\begin{cases} \alpha_1(\theta, A_1) = \alpha_1(\theta, |A_1|^2), \\ \alpha_2(\theta, A_1) = \alpha_2(\theta, |A_1|^2), \\ \alpha_1 \text{ is bounded and } |\alpha_1| \leq c_1, \quad \text{where } c_1 \text{ is a strictly positive constant.} \end{cases} \tag{9}$$

Our first result is the following:

Proposition 2.1 For any fluid of complexity 2 defined by (5) and satisfying (8), one has

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |u|^2 \mathrm{d}x + \frac{1}{2} \int_0^{|A_1|^2} \alpha_1(\theta, \xi) \mathrm{d}\xi + \int_\Omega \mu |A_1|^2 \mathrm{d}x + \int_\Omega (\alpha_1 + \alpha_2) \mathrm{tr} A_1^3 \mathrm{d}x = 0. \tag{10}$$

Let $N(\cdot) : \mathbf{R}^+ \mapsto \mathbf{R}$ be the function definded by

$$N(t) = -\frac{1}{2} \int_\Omega \Big(\int_0^{|A_1|^2} \alpha_1(\theta, \xi) \mathrm{d}\xi \Big) \mathrm{d}x - \int_\Omega |u|^2 \mathrm{d}x. \tag{11}$$

Then we have the following result:

Proposition 2.2 Let u be a velocity field satisfying (10). Suppose also that

$$\int_0^z \alpha_1(\theta, \xi) \mathrm{d}\xi < -2C_\Omega (1 + \eta) z, \quad \forall\, z > 0, \quad \text{for some } \ \eta > 0. \tag{12}$$

Then, for every Ω and any field $u(\cdot,0)$, one has

$$N(t) \geq 0, \quad \forall\, t \geq 0. \tag{13}$$

Suppose moreover, that

$$u(\cdot,0) \neq 0, \quad \text{on} \quad \Omega, \tag{14}$$

and

$$\alpha_2 \ \text{ is bounded in } \ \Omega \times [0,+\infty), \tag{15}$$

and also that

$$|\alpha_1 + \alpha_2| > \varepsilon \ \text{ on } \ \Omega \times [0,+\infty), \quad \text{for arbitrary } \ \varepsilon > 0. \tag{16}$$

Then

$$(a) \quad N(0) > 0, \qquad (b) \quad N(t) > 0 \quad \text{on} \quad [0,+\infty). \tag{17}$$

In the same range of assertions, we have

Theorem 2.3 Let u be a velocity field satisfying (10). Assume that (12), (14) and (16) are fulfilled. Moreover, if,

$$\alpha_2(\cdot,t) \in L^{3/2}(\Omega), \quad \forall\, t \geq 0.$$

Then, for a sufficiently large viscosity and for any domain Ω, one has

$$\forall\, M > 0, \quad \exists\, t_M > 0 \quad \text{such that} \quad \int_\Omega |\alpha_1 + \alpha_2|\, |A_1|^3(x,t_M)\mathrm{d}x > M.$$

Corollary 2.4 Let u be a velocity field satisfying (10). Assume that (15) and (16) hold. If the upper bound b of α_1 is strictly negative on $\Omega \times [0,+\infty)$, then in any arbitrary flow, for a sufficiently large viscosity and in a rather small domain Ω, one has

$$\forall\, M > 0, \quad \exists\, t_M > 0 \quad \text{such that} \quad \int_\Omega |A_1|^3(x,t_M)\mathrm{d}x > M.$$

Proposition 2.5 Let again u be a velocity field satisfying (10). If (10), (14), (15) and (16) hold true, then, for a sufficiently large viscosity and for any Ω, one has

$$\liminf_{t\to\infty} \left(\sup_\Omega |A_1|(t) \right) \neq 0.$$

Theorem 2.6 Let again u be a velocity field satisfying (10). Suppose that (12) and (14) are satisfied. If

$$\alpha_1 + \alpha_2 \equiv 0, \quad \text{on} \quad \Omega \times [0,+\infty), \tag{18}$$

then

$$\|u(t)\|^2 \geq [N(0)\exp(\lambda t)]c_1, \quad \forall\, \lambda \quad \text{such that} \quad 0 < \lambda < \frac{2\mu}{c_1}.$$

Corollary 2.7 Under hypotheses (14) and (18) and if the upper bound b of α_1 is strictly negative on $\Omega \times [0,+\infty)$, then

$$\|u(t)\|^2 \geq \frac{[N(0)\exp(\lambda t)]}{-b}, \quad \forall\, \lambda \quad \text{such that} \quad 0 < \lambda < \frac{2\mu}{-b}.$$

Our last result is the next proposition.

Theorem 2.8 Let us consider a fluid defined by (5) and system (8) with initial data satisfying (14). If $\Omega \subset \mathbf{R}^2$, then, independently of the choice of the sign of $\alpha_1+\alpha_2$ (i.e., either $|\alpha_1+\alpha_2|>\varepsilon$ or $\alpha_1+\alpha_2\equiv 0$), one has

$$\|u(t)\|^2 \geq [N(0)\exp(\lambda t)]c_1, \quad \forall\, \lambda \quad \text{such that} \quad 0<\lambda<\frac{2\mu}{c_1}.$$

3 Proofs of the results

3.1 Proof of Proposition 2.1

It is sufficient to take the scalar product of the first equation in (8) with u, to integrate over Ω and to use the diveregence theorem to obtain the following energy equation

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2\mathrm{d}x + \int_\Omega T\cdot L\mathrm{d}x = \int_{\partial\Omega} u\cdot Tn\mathrm{d}\sigma + \int_\Omega uf\mathrm{d}x. \tag{19}$$

Since f derives from a potential, with the help of (5), inequality (19) becomes

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2\mathrm{d}x + \int_\Omega T\cdot L\mathrm{d}x = 0, \quad \forall\, t\geq 0. \tag{20}$$

Making use of (5) and taking into account the symmetry of the stress tensor T, one gets

$$T\cdot L = \frac{1}{2}T\cdot A_1 = -pL + \frac{1}{2}\Big(\mu|A_1|^2 + \frac{1}{2}\alpha_1(\theta,|A_1|^2)\frac{\mathrm{d}}{\mathrm{d}t}|A_1|^2 + (\alpha_1+\alpha_2)\mathrm{tr}A_1^3\Big), \tag{21}$$

where we used the fact that

$$A_2\cdot A_1 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|A_1|^2 + \mathrm{tr}A_1^3.$$

Substituting (21) in (20) and using again the fact that div $u=0$, gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2\mathrm{d}x + \frac{1}{2}\int_0^{|A_1|^2}\alpha_1(\theta,\xi)\mathrm{d}\xi + \int_\Omega \mu|A_1|^2\mathrm{d}x + \int_\Omega(\alpha_1+\alpha_2)\mathrm{tr}A_1^3\mathrm{d}x = 0,$$

and this concludes the proof of Proposition 2.1.

3.2 Proof of Proposition 2.2

3.2.1 Proof of inequality (13)

From (11) we can write

$$N(t) = -\frac{1}{2}\int_\Omega\int_0^{|A_1|^2}\alpha_1(\theta,\xi)\mathrm{d}\xi\mathrm{d}x - C_\Omega\int_\Omega |A_1|^2\mathrm{d}x + C_\Omega\int_\Omega |A_1|^2\mathrm{d}x - \int_\Omega |u|^2\mathrm{d}x. \tag{22}$$

Observe now that

$$\int_\Omega |A_1|^2\mathrm{d}x = 2\int_\Omega |\nabla u|^2\mathrm{d}x,$$

since div $u=0$ and $u(\cdot,t)=0$ on $\partial\Omega$. Applying the Poincaré inequality yields

$$N(t) \geq -\frac{1}{2}\int_\Omega\int_0^{|A_1|^2}\alpha_1(\theta,\xi)\mathrm{d}\xi\mathrm{d}x - C_\Omega\int_\Omega |A_1|^2\mathrm{d}x. \tag{23}$$

Due to hypothesis (12), one can estimate again the second term in the right hand side of (23) to get

$$N(t) \geq 2(1+\eta)C_\Omega \int_\Omega |A_1|^2 \mathrm{d}x - C_\Omega \int_\Omega |A_1|^2 \mathrm{d}x,$$

so that there is some $\zeta > 0$, such that

$$N(t) \geq \zeta C_\Omega \int_\Omega |A_1|^2 \mathrm{d}x, \quad \forall\, t \geq 0,$$

and this ends the proof of (13).

3.2.2 Proof of inequality (17)(a)

Since $u(x,0) \neq 0$ on Ω, there exists $x_0 \in \Omega$ such that $u_0(x_0,0) \neq 0$. The continuity with respect to x of the velocity field u, implies the existence of a neighborhood Ω_0 of x_0, $\Omega_0 \subset \Omega$ of non zero measure, such that $u(x,0) \neq 0$ for $x \in \Omega_0$. Consequently, there also exists a subset of Ω_0 of non zero measure, on which $A_1(\cdot,0) \neq 0$ too. Indeed, if this is not true, as Ω is smooth enough and connected, the smoothness of $\partial\Omega$ and the adherence condition of the fluid to the boundary of Ω, would imply

$$u(\cdot,0) \in C(\overline{\Omega}) \quad \text{and} \quad u \equiv 0 \quad \text{on} \quad \Omega,$$

which is in contradiction with hypothesis (14). We deduce that we must have

$$\eta C_\Omega \int_\Omega |A_1|^2 \mathrm{d}x > 0, \quad \text{for} \ \ t = 0,$$

when $N(0) > 0$.

3.2.2 Proof of inequality (17)(b)

The proof is based on the following result, due to Fosdick et Rajagopal[9].

Lemma 3.1 For every A in $\mathcal{T}_s^0$ and for any $\alpha \in \mathbf{R}$, one has

$$-\frac{|\alpha|}{\sqrt{6}}|A|^3 \leq \alpha \mathrm{tr} A^3 \leq \frac{|\alpha|}{\sqrt{6}}|A|^3.$$

Notice that by definition and thanks to the fact that div $u = 0$, one has $A_1 \in \mathcal{T}_s^0$. Now, let us consider the set

$$E = \{t \in [0,+\infty) \mid N(t) = 0\}.$$

If we suppose that it is not void, then it is easily seen that

$$N(t) = 0 \Longleftrightarrow A_1(x,t) \equiv 0 \quad \text{on} \quad \Omega. \tag{24}$$

But

$$\int_\Omega |A_1|^2 \mathrm{d}x = 2\int_\Omega |\nabla u|^2 \mathrm{d}x,$$

so that $u(\cdot,t) \equiv 0$ on Ω for any $t > 0$, this being a consequence of the hypothesis that the motion is smooth, of the fact that Ω is connected and of the boundary condition $u = 0$ on $\partial\Omega$. Consequently, $N(t) = 0$ for any $t > 0$.

On the other hand, if t_0 is the lower bound of the set E, then $N(t_0) = 0$. It follows that

$$A_1(\cdot,t_0) \equiv 0 \quad \text{on} \quad \Omega.$$

As $u=0$ on $\partial\Omega$, the smoothness of u and of $\partial\Omega$ imply that $A_1(\cdot,t)$ belongs to the class $C(\overline{\Omega})$. Consequently, $A_1(\cdot,t)$ admits a maximum on $\overline{\Omega}$, maximum given by

$$\sup_{\Omega} A_1(x,t).$$

Now, let c_2 be such that $|\alpha_1+\alpha_2|<c_2$ on $\Omega\times[0,+\infty)$. By using Lemma 3.1, one can estimate the following function

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t)=\int_{\Omega}\mu|A_1|^2\mathrm{d}x+\int_{\Omega}(\alpha_1+\alpha_2)\mathrm{tr}A_1^3\mathrm{d}x. \tag{25}$$

Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t)\geq\mu\int_{\Omega}|A_1|^2\mathrm{d}x-\frac{c_2}{\sqrt{6}}\int_{\Omega}|A_1|^3\mathrm{d}x,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t)\geq\left[\mu-\frac{c_2}{\sqrt{6}}\sup_{\Omega}|A_1|(t)\right]\int_{\Omega}|A_1|^2\mathrm{d}x. \tag{26}$$

But t_0 is the smallest time for which $A_1(\cdot,t_0)\equiv 0$ on Ω (see (24)). As a consequenc $A_1(\cdot,t)\neq 0$ on Ω for any t such that $0<t<t_0$. It follows that for any t with $0<t<t_0$, there exists $\Omega_t\subset\Omega$ of non zero measure, on which $A_1(x,t)\neq 0$.

Moreover, the continuity of $A_1(\cdot,\cdot)$ allows us to find an interval $[t_0-\delta,t_0)$ such that

$$0<\sup_{\Omega}|A_1|(t)<\frac{\mu\sqrt{6}}{c_2},\quad\forall\, t\in[t_0-\delta,t_0). \tag{27}$$

Recalling that

$$A_1(x,t)\neq 0,\quad\forall\, x\in\bigcup_{t\in[t_0-\delta,t_0)}\Omega_t,$$

we get

$$\int_{\Omega}|A_1|^2\mathrm{d}x>0.$$

Therefore, by making use of (26) and (27),

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t)>0,\quad\forall\, t\in[t_0-\delta,t_0).$$

This implies that the function N is strictly incresing on $[t_0-\delta,t_0)$. Since N is strictly positive on $[t_0-\delta,t_0)$, one has

$$\lim_{t\to t_0}N(t)>0.$$

Due to the continuity of N, one has $N(t_0)>0$, which is contradictory to $N(t_0)=0$. So the set E is empty, and therefore $N(t)$ is strictly positive for $t\geq 0$.

3.3 Proof of Theorem 2.3

Let λ be an arbitrary real number. From (25), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t)-\lambda N(t)\geq\left(\mu-\frac{\lambda c_1}{2}\right)\int_{\Omega}|A_1|^2\mathrm{d}x-\int\Omega\frac{|\alpha_1+\alpha_2|}{\sqrt{6}}|A_1|^3\mathrm{d}x+\lambda\int_{\Omega}|u|^2\mathrm{d}x. \tag{28}$$

Choosing λ such that

$$0<\lambda<\frac{2\mu}{c_1}, \tag{29}$$

inequality (28) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t)-\lambda N(t)\geq\int_\Omega\frac{|\alpha_1+\alpha_2|}{\sqrt{6}}|A_1|^3\mathrm{d}x. \tag{30}$$

Suppose now by absurd, that

$$\exists\, M>0 \quad \text{such that } \ \forall\, t\geq 0, \quad \int_\Omega|\alpha_1+\alpha_2|\,|A_1|^3\mathrm{d}x\leq M. \tag{31}$$

By virtue of hypothesis (16),

$$\int_\Omega|A_1|^3\mathrm{d}x\leq\frac{M}{\varepsilon}, \quad \forall\, t\geq 0. \tag{32}$$

Replacing (31) into inequality (30), yields

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t)-\lambda N(t)\geq-\frac{M}{\sqrt{6}}.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(N(t)\exp(-\lambda t)\big)\geq-\frac{M}{\sqrt{6}}\exp(-\lambda t).$$

Therefore

$$N(t)\geq\Big(N(0)-\frac{M}{\lambda\sqrt{6}}\Big)\exp(\lambda t)+\frac{M}{\lambda\sqrt{6}},$$

ans so,

$$N(t)\geq\Big(N(0)-\frac{M}{\lambda\sqrt{6}}\Big)\exp(\lambda t).$$

Since $N(0)>0$, choosing μ sufficiently large, according to (29), one can also take λ sufficiently large, in order to get

$$N(0)-\frac{M}{\lambda\sqrt{6}}>0.$$

This implies an exponential growth of $N(t)$ when $t\to+\infty$. Then from definition (11) of N, we have

$$\int_\Omega\int_0^{|A_1|^2}\alpha_1(\theta,\xi)\mathrm{d}\xi\mathrm{d}x\to\infty \quad \text{for} \quad t\to+\infty, \tag{33}$$

whence, by Holder inequality and since α_1 is bounded

$$\int_\Omega|A_1|^3\mathrm{d}x\to\infty \quad \text{for } \ t\to+\infty.$$

But this result is in contradiction with (32). Therefore, (31) does not hold and this ends the proof.

3.4 Proof of Corollary 2.4

Choose Ω in order to have

$$C_\Omega<\frac{-b}{2(1+\eta)} \quad \text{for an arbitrary} \quad \eta>0.$$

Hypothesis (12) is then satisfied and the corollary is immediate.

3.5 Proof of Proposition 2.5

Proposition 2.2 states that $N(0) > 0$ on $[0, +\infty)$. Then definition (11) leads to the following inequality

$$\int_{\Omega} \int_0^{|A_1|^2} \alpha_1(\theta, \xi) \mathrm{d}\xi \mathrm{d}x < 0, \quad \forall\, t \geq 0.$$

It follows that there exists $\Omega_0 \subset \Omega$ of non zero measure, such that

$$\int_{\Omega} \int_0^{|A_1|^2} \alpha_1(\theta, \xi) \mathrm{d}\xi \mathrm{d}x \neq 0, \quad \forall\, t \geq 0,$$

hence,

$$|A_1|(x, t) \neq 0 \quad \text{on} \quad \Omega_0, \quad \forall\, t \geq 0.$$

Consequently,

$$\sup_{\Omega} |A_1|(t) \geq 0, \quad \forall\, t \geq 0. \tag{34}$$

We now prove that the strict positivity of

$$\sup_{\Omega} |A_1|(t)$$

is conserved as $t \to +\infty$. Indeed, in view of inequality (34), only two cases can occur.

Case 1 For any $T \in \mathbf{R}^+$, there exists $\tau \geq T$ such that

$$\sup_{\Omega} |A_1|(\tau) \geq \frac{\mu\sqrt{6}}{c_2},$$

and so,

$$\liminf_{t \to \infty} \sup_{\Omega} |A_1|(t) \neq 0.$$

Case 2 There exists $T > 0$ such that for any $t > T$, one has

$$0 < \sup_{\Omega} |A_1|(t) < \frac{\mu\sqrt{6}}{c_2}.$$

Then, by (27) et (29), one has

$$\liminf_{t \to \infty} \int_{\Omega} \Big| \int_0^{|A_1|^2} \alpha_1(\theta, \xi) \mathrm{d}\xi \Big| \mathrm{d}x - \int_{\Omega} |u|^2 \mathrm{d}x > 0.$$

Since α_1 is bounded,

$$\liminf_{t \to \infty} \int_{\Omega} |A_1|^2 \mathrm{d}x > 0.$$

On the other hand, as $L^{\infty}(\Omega) \subset L^2(\Omega)$ with continuous imbedding, it follows that

$$0 < \liminf_{t \to +\infty} \int_{\Omega} |A_1|^2 \mathrm{d}x < k \liminf_{t \to +\infty} \sup_{\Omega} |A_1|(t),$$

where k is a positive constant. Consequently,

$$\liminf_{t \to +\infty} \sup_{\Omega} |A_1|(t) > 0,$$

and the proof of Proposition 2.5 is now complete.

3.6 Proof of Theorem 2.6

From (25) and hypothesis (18), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \mu \int_\Omega |A_1|^2 \mathrm{d}x,$$

so that the function

$$\mu \int_\Omega |A_1|^2 \mathrm{d}x$$

is increasing on $[0, +\infty)$. Let λ be an arbitrary real number and introduce the following function

$$F(t) = \frac{\mathrm{d}}{\mathrm{d}t}N(t) - \lambda N(t).$$

Then, recalling definition 11, it is immediately seen that

$$F(t) = \mu \int_\Omega |A_1|^2 \mathrm{d}x - \frac{\lambda}{2} \int_\Omega \Big| \int_0^{|A_1|^2} \alpha_1(\theta, \xi) \mathrm{d}\xi \Big| \mathrm{d}x + \lambda \int_\Omega |u|^2 \mathrm{d}x.$$

Since the function α_1 is bounded, if we choose λ in the interval $[0, \frac{2\mu}{c_1}]$, one has

$$F(t) \geq 0, \quad \forall\, t \geq 0,$$

and therefore,

$$N(t) \geq N(0) \exp(\lambda t), \quad \forall\, t \geq 0.$$

By Proposition 2.2, $N(0) > 0$, and so the function N has an exponential growth.

From the expression of N, we get the following inequality

$$\frac{c_1}{2} \int_\Omega |A_1|^2 \mathrm{d}x \geq N(0) \exp(\lambda t), \quad \forall\, t \geq 0.$$

But

$$\int_\Omega |A_1|^2 \mathrm{d}x = 2 \int_\Omega |\nabla u|^2 \mathrm{d}x,$$

so that

$$\|u\|^2 \geq \frac{N(0)}{c_1} \exp(\lambda t), \quad \forall\, t \geq 0.$$

3.7 Proof of Theorem 2.7

The result is a simple application of Theorem 2.6.

3.8 Proof of Theorem 2.8

Applying Hamilton-Cayley theorem, we get

$$A^2 + I \det A = 0.$$

Multiplying by A and taking into account that $\mathrm{tr}A = 0$, it follows that $\mathrm{tr}A^3 = 0$. Then, the theorem follows from all the preceding results.

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关于一些非牛顿流体

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摘　要: 我们考虑了满足Clausius-Puhem不等式的一类二维Rivlin Ericksen流体。对于严格孤立的流体，我们证明了如果处于均衡时Helmhdtz自由能量非最小，则在一定条件下所有原始扰动不仅会放大而且会增加。这些结果推广了Fosclish Rajagopal在1982年得到的二维Rivlin Ericksen流体的一种特例—二等级流体的相关结果。

关键词: 非Newton流体；可微流体；二等级流体；Clausius-Douhem不等式；Helmholty自由能量